Math 261B Tues. 10/27

GL_n, SL_n

$$\mathcal{O}_{\mathbb{Z}}(GL_n) = \mathbb{Z}[a_{11}, \ldots, a_{nn}, z] / (z \det(A) - 1)$$
$$= \mathbb{Z}[a_{11}, \ldots, a_{nn}, \det(A)^{-1}]$$

is a Hopf algebra over $\mathbb{Z}$

$$\mathcal{O}_{\mathbb{Z}}(SL_n) = \mathbb{Z}[a_{11}, \ldots, a_{nn}] / (\det A - 1)$$

$$\{\mathcal{O}_{\mathbb{Z}}(GL_n) \to R\} = GL_n(R)$$

SO_N $\quad \mathbb{Z}[a_{11}, \ldots, a_{NN}] / (Q(Ax) = Q(x), \det A - 1)$ (or something)

$$Q(x) = x_1 x_N + x_2 x_{N-1} + \cdots +$$

$$PGL_2 = PSL_2 = SL_2 / \{\pm I\}$$

$$\mathcal{O}_{\mathbb{Z}}(SL_2) = \mathbb{Z}[a,b,c,d] / (ad - bc - 1)$$

is $\mathbb{Z}/2\mathbb{Z}$ graded

$\{\pm 1\}$ $\quad V = V_0 \oplus V_1$

V_0: trivial; V_1: -1 acts as -1

$\pm I \subset \mathcal{O}_{\mathbb{Z}}(SL_2)$ $\quad \mathcal{O}_{\mathbb{Z}}(SL_2) \to \mathcal{O}_{\mathbb{Z}}[\{\pm 1\}]$

$$a \mapsto \delta_e - \delta_s$$

$$\mathcal{O}_{\mathbb{Z}}[\{\pm 1\}] = \underset{\mathbb{Z}\delta_e \oplus \mathbb{Z}\delta_s}{\underset{\parallel}{\mathbb{Z}\times\mathbb{Z}}} = \{\{\pm 1\} \to \mathbb{Z}\} \qquad \mathbb{Z}\cdot\{\pm 1\} = \mathbb{Z}e \oplus \mathbb{Z}s$$

$$\{\pm 1\} \leftarrow \{e, s\}$$

$$\delta_e^2 = \delta_e \quad \delta_s^2 = \delta_s$$

$$\delta_e\delta_s = \delta_s\delta_e = 0$$

$$1 = \delta_e + \delta_s$$

$$\Delta\delta_s = \delta_e \otimes \delta_s + \delta_s \otimes \delta_e$$

$$\Delta\delta_e = \delta_e \otimes \delta_e + \delta_s \otimes \delta_s$$

$$s^2 = e$$

$$s^2 = 1 \qquad e = 1$$

$$\Delta e = e \otimes e, \quad \Delta s = s \otimes s$$

$$\varepsilon(e) = \varepsilon(s) = 1$$

$$\{\mathcal{O}_{\mathbb{Z}}[\{\pm 1\}] \to K\} = \{\pm 1\} \qquad K_e, K_s$$

$\mathcal{O}_{\mathbb{Z}}[\{\pm 1\}] \to R$ = {pairs of orth. idempotents in R}

$$\operatorname{Spec}(R) \to \operatorname{Spec}(\mathcal{O}_{\mathbb{Z}}[\{\pm 1\}]) \qquad R = S \times S' \qquad \operatorname{Spec}(R) = \operatorname{Spec}(S) \sqcup \operatorname{Spec}(S')$$

$$1, 0$$
$$0, 1$$

$\mathcal{O}_{\mathbb{Z}}[\{\pm 1\}]$ comodules = $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups.

$$\begin{array}{ccc} & V & \\ \swarrow & & \searrow \\ V_1 & & V_{-1} \\ \downarrow & & \swarrow \\ & V \oplus V & \end{array}$$

$\mathbb{Z}$-module $\quad V \xrightarrow{\rho} V \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}[\{\pm 1\}] \cong V \oplus V$

$$V \otimes \delta_e \oplus V \otimes \delta_s$$

$$V = V_1 \oplus V_{-1}$$

$v \in V_1 \quad : \quad \rho(v) = v \otimes \delta_e + v \otimes \delta_s$

$v \in V_{-1} \quad \rho(v) = v \otimes \delta_e - v \otimes \delta_s$

$$V \to V \otimes \mathcal{O}_{\mathbb{Z}}(SL_2)$$
$$\downarrow$$
$$V \otimes \mathcal{O}_{\mathbb{Z}}[\{\pm I\}]$$

$\mathcal{O}_{\mathbb{Z}}(SL_2) \to \mathcal{O}_{\mathbb{Z}}[\{\pm I\}] \quad \leftarrow\cdot\cdot\to \quad \pm I \subset SL_2$

$a \mapsto \delta_e - \delta_s$

$b \mapsto 0$

$c \mapsto 0$

$d \mapsto \delta_e - \delta_s$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\pm I) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$s \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

$$ad - bc \mapsto (\delta_e - \delta_s)^2 = \delta_e^2 + \delta_s^2 = \delta_e + \delta_s = 1$$

$$\mathcal{O}_{\mathbb{Z}}(PSL_2) = (\mathcal{O}_{\mathbb{Z}}(SL_2))_{2\mathbb{Z}} = \left(\mathbb{Z}[a,b,c,d] / (\det(A) - 1)\right)_{2\mathbb{Z}}$$

$\shortparallel$

$\mathcal{O}_{\mathbb{Z}}(PGL_2) \qquad = \mathbb{Z}[a^2, ab, \dots] / (ad - bc - 1,$ tautological relations $)$

$(a^2)(b^2) = (ab)^2$

$$\underline{PSL_2}(R) \overset{?}{=} PSL_2(R) \neq SL_2(R) / \{\pm I\}$$

$\shortparallel$

$\mathrm{Hom}(\mathcal{O}_{\mathbb{Z}}(PSL_2), R)$

Chevalley $\mathbb{Z}$ forms

Reductive group data for $G \supset B \supset T$:

$X = X(T)$, X^*, $R = R_+ \sqcup R_-$, $R^\vee$, $(W, \dots)$

Look at char 0 case: ($K=$)

Standard h.w. irr. modules V^λ

have canonical $\mathbb{Z}$ forms $V^\lambda \supseteq \mathbb{Z}^N$ $\quad N = \dim(V^\lambda)$ $\quad K = \mathbb{C}$

s.t. $V^\lambda = K \cdot \mathbb{Z}^N$, compatible with weight spaces

$$V^\lambda \overset{?}{=} \bigoplus_{\mu \in X} (V^\lambda)_\mu \qquad (V^\lambda)_\mu = \mathbb{Z}^{m_\mu} \qquad m_\mu = \dim (V^\lambda)_\mu$$

$$\mathbb{Z}^N = \bigoplus_\mu \mathbb{Z}^{m_\mu}$$

$$\mathcal{O}_\mathbb{Z}(T) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \qquad n = \dim(T)$$

$$\Delta x_i = x_i \otimes x_i \qquad v \overset{\rho}{\longmapsto} v \otimes x^\lambda$$

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad T \subset G$$

$$\mathcal{O}_\mathbb{Z}(G) \to \mathcal{O}_\mathbb{Z}(T)$$

Simple roots α_i, pick $E_i \in \mathfrak{g}_{\alpha_i}$ $\quad F_i \in \mathfrak{g}_{-\alpha_i}$ s.t.

E_i, $H_i = [E_i, F_i]$, F_i are "standard" in $\mathfrak{g}_{\alpha_i} \oplus \mathfrak{h}_i \oplus \mathfrak{g}_{-\alpha_i} = \mathfrak{sl}_2$

sl_2 : $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $[E, F] = H$, $[H, E] = 2E$, $[H, F] = -2F$

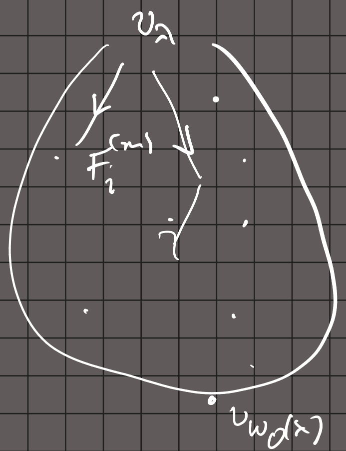
$\cap$
T

E_i's, F_i's unique up to T conjugation.

$\mathfrak{g} \hookrightarrow \mathcal{O}(G)^*$

$\mathbb{Z}^N \subset V^\lambda$ is closed under action of $E_i^{(m)} = E_i^m / m!$

In fact, can take $\mathbb{Z}^N$ = closure under $F_i^{(m)}$ of a H.W. v_λ.



λ

V^λ

$F_{i_1}^{(m_1)} \cdots F_{i_k}^{(m_k)} v_\lambda$ $\qquad v_\lambda$

$\uparrow$

take their $\mathbb{Z}$ span

This is also closed under the $E_i^{(m)}$

$\mathcal{O}_{\mathbb{Z}}(G) \subset \mathcal{O}_K(G)$ subring gen. by matrix coefficients in the Chevalley $\mathbb{Z}$ forms.

<u>Examples</u> $V = \Lambda^d K^n$ $\quad (G = GL_n,\ SL_n,\ \dots)$

$= V_{(1,\dots,1,0,\dots,0)}$

Basis of wedge monomials $e_{i_1} \wedge \cdots \wedge e_{i_d}$ $\quad i_1 < \cdots < i_d \subset [1,n]$

$v_\lambda = e_1 \wedge \cdots \wedge e_d$

K^n : defining rep.

$G = GL_n$

$\mathfrak{g} = \mathfrak{gl}_n$ = matrices $\begin{pmatrix} \ddots & & \\ & & \\ & & \end{pmatrix}$

E_i is $\begin{array}{cc} & i+1 \\ & \downarrow \\ \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} & \leftarrow i \end{array}$

F_i is $\begin{array}{cc} i & \\ \downarrow & \\ \begin{pmatrix} \ddots & & \\ 1 & \ddots & \\ & & \ddots \end{pmatrix} & \leftarrow i+1 \end{array}$ $\quad \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & a\ b & \\ & & c\ d & \\ & & & \ddots \end{pmatrix}$ $\quad \begin{pmatrix} 0 & & \\ & \begin{matrix} a & b \\ c & -a \end{matrix} & \\ & & 0 \end{pmatrix}$

E_i is $e_{i+1} \mapsto e_i$ other $e_j \mapsto 0$

F_i is $e_i \mapsto e_{i+1}$ others $\mapsto 0$

F_i $e_{i_1} \wedge \cdots \wedge e_{i_d}$ is $e_i \mapsto e_{i+1}$

$F_i^2 = 0$ $F_i^{(m)} = F_i^m / m!$

Wedge basis is a Chevalley basis.